Better Than a 50-50 Proposition

by Jim Stein, Emeritus Professor of Mathematics (CSULB)

The term Emeritus comes from the Latin

‘ex’ – meaning ‘out’

and

‘meritus’ – meaning ‘ought to be’

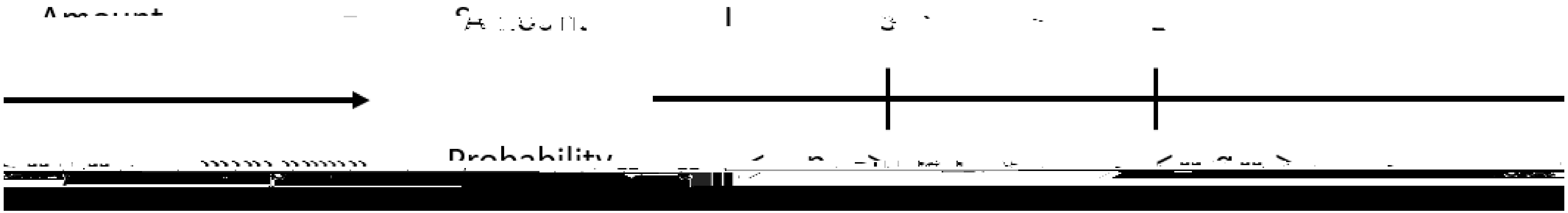
Blackwell's Bet

You are shown a red envelope and a blue envelope, each containing a different amount of money. You are allowed to open one envelope and count the money in it. You can now either keep the money in the envelope, or switch your choice and keep the amount of money in the unopened envelope. Your goal, of course, is to end up with the larger amount of money.

Yes, it DOES sound a little like the Monty Hall Problem – but although there's a right thing to do, like the MHP, the strategy is considerably different.

Blackwell's brilliant suggestion is to pick a random number and compare it with the amount of money in the opened envelope. If the random number is less than that amount, keep the money – if it's bigger, take the money in the other envelope.

The math is really simple. Let S denote the smaller amount of money, L the larger. Let R denote the random number. Let p denote the probability that $R < S$, and let q denote the probability that $R > L$.



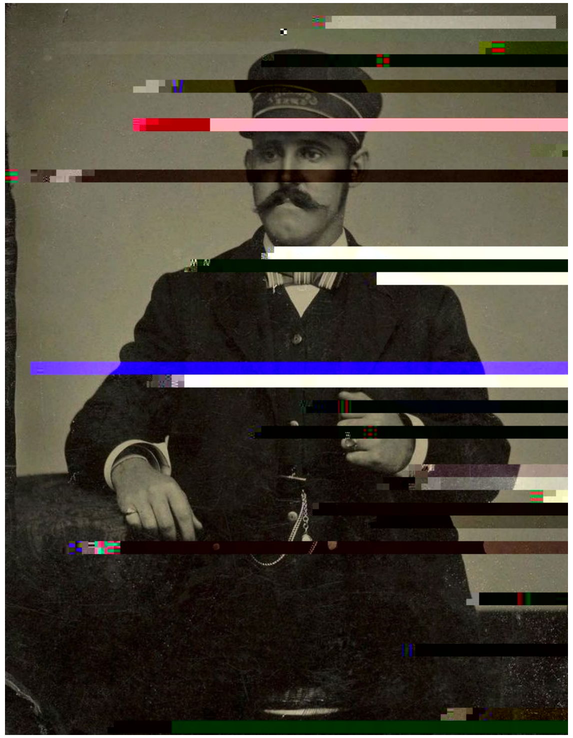
With probability ½, we opened the envelope with the smaller amount of money. We make the winning decision to switch if $R > S$, which occurs with probability 1-p. The combined probability of opening the envelope with the smaller amount of money and guessing correctly to switch is ½ (1-p).

With probability ½, we opened the envelope with the larger amount of money. We make the winning decision to keep the money if $R < L$, which occurs with probability 1-q. The combined probability of opening the envelope with the larger amount of money and guessing correctly to keep it is ½ (1-q).

The probability of ending up with the larger amount is ½(1-p) + ½(1-q) = ½ + ½(1-(p+q)). This is ½ plus half the probability that the random number R falls in the gap between S and L.

A Stop at Willoughby (from *The Twilight Zone*)

You board an East-West railroad line whose movement is a random walk governed by the flip of a fair coin. If the coin lands heads, the train moves East to the next station, if it lands tails, it moves West to the next station. You fall asleep, and when you wake up you find yourself in a railroad car that looks like it came from the 1890s.



Now let's assume we're actually at Willoughby, and we're going to try to guess which direction the train will head by using the same technique.

Let R denote a random location on the railroad line. We'll guess that the train will head towards that location. Let p denote the probability that R is West of Willoughby. Then 1-p denotes the probability that R is East of Willoughby.

[redacted]

With probability ½, the coin flip directs us West, and we guess correctly that we will head West with probability p.

With probability ½, the coin flip directs us East, and we guess correctly that we will head East with probability 1- p.

The probability of correctly guessing which way the train is going is $\frac{1}{2}p + \frac{1}{2}(1-p) = \frac{1}{2}$

We need someone to put these two results into the cosmic perspective that they deserve, and who better than …

Yoda!!

Yoda speaks.

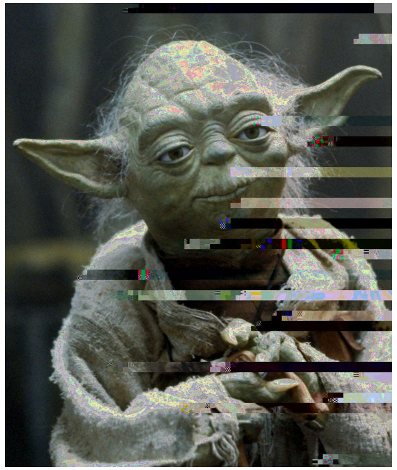
"Learned you have, young Skywalker, all you need about the Force.

Now learn you must about GLOW, the Great Lesson of Willoughby".

You are more likely to know where you are if you know where you are going than you are to know where you are going if you know where you are.

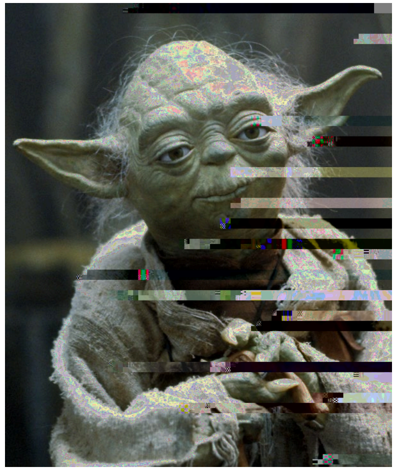
But what would happen, O knowledgeable one, if the train were equally likely to be

And what if there are only a finite number of stations?

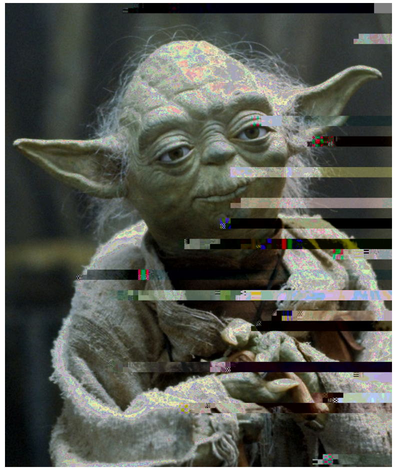


The limiting distribution of a Markov chain in such a situation shows that the train is only half as likely to be in the easternmost or westernmost station as it is to be in any other station.

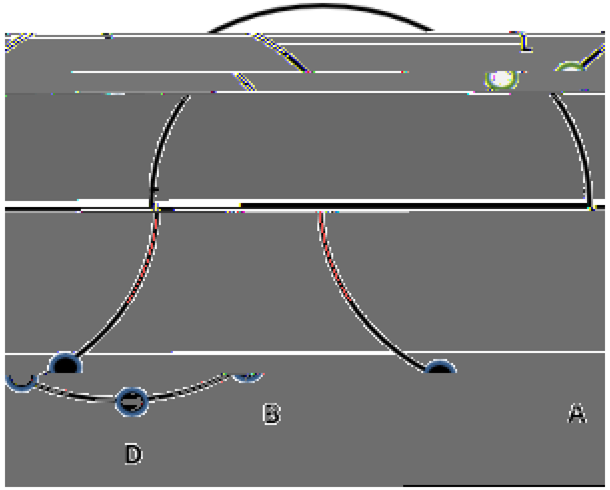
But suppose, O wise one, that there were a finite number of stations arranged on a circle? Instead of moving East or West, the train would move clockwise or counterclockwise. Wouldn't this limiting distribution of which you speak result in the train being equally likely to be at any station?



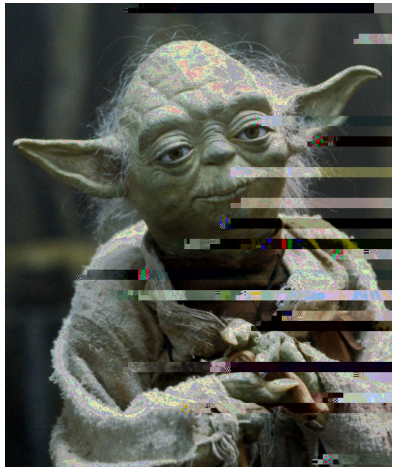
Truly have you acquired enlightenment, young Skywalker. Moreover, each station is equally likely to be the next destination of the train. The probability of any station being the present location of the train is simply the reciprocal of the number of stations, as is the probability of any station being the next destination of the train.

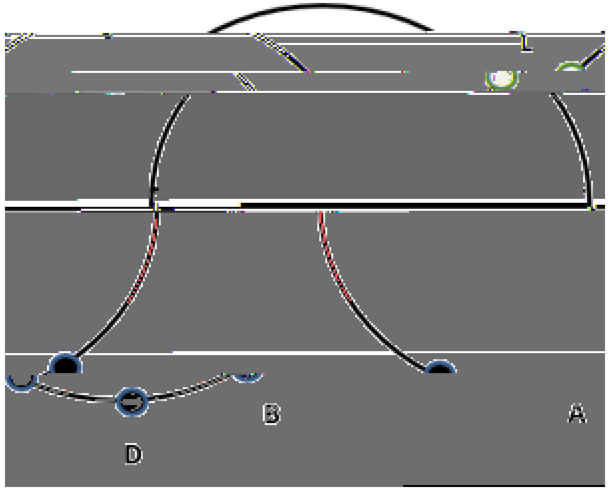


But beware, young Skywalker! The situation is not as simple as one might suspect from an examination of the linear case. There is still the matter of deciding which way the train will go when it leaves a station. In order to do this, we must first position lights midway between each adjacent pair of stations.



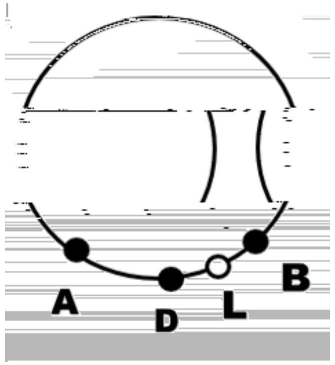
Suppose that light L is on, and D represents the next destination of the train. We are once again at stations A or B with equal probability. We choose a uniformly distributed random location R on the track, and guess that the train will head along the arc of the circle for which we

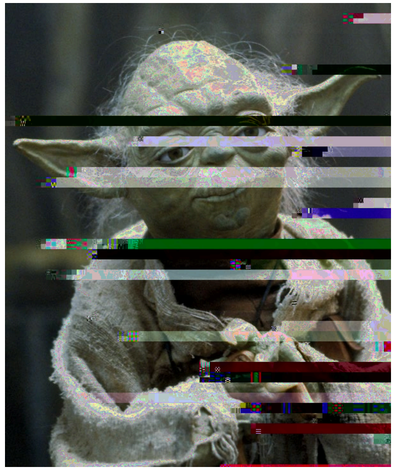




Repeated for convenience

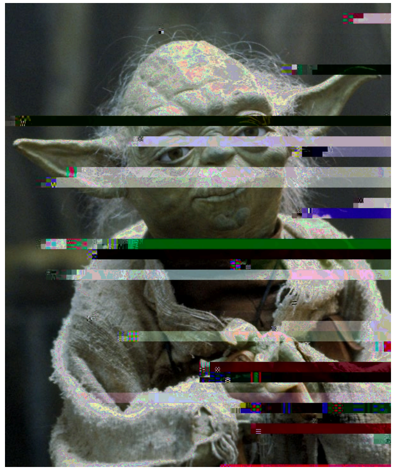
It seems, O guardian of the flame of wisdom, that there might be a difference in the probability of guessing correctly if the light L is located on the minor arc between A and B.





Realized you have, young Luke, that there is more to being a Jedi than the use of the light saber. In the diagram above, we are at station A with probability ½. The probability that we will correctly guess the direction of the train is the ratio of arc AL to the entire circle, and so the probability of being at A and guessing correctly is ½ arc AL.

So it seems that one must take pains to ensure that the minor arc between stations A and B must not contain the light.



Only if one wishes that the probability of guessing correctly prior to EVERY move is greater than ½ -- and I doubt that this can be achieved.

But if one is willing to settle for the probability that ON AVERAGE one guesses correctly more than half the time, I believe there is a way. When the walk has lasted sufficiently long that the train's location is close to the limiting distribution, turn on the light diametrically opposite the train's location – or as close as possible to this -- and leave it on. At any subsequent time, the train will be more likely to be at or near its location when the light was turned on. It will be less likely that the minor arc between stations A and B will contain the light, and my calculations show this is good enough.

The Willoughby Paradox

Assume that the train is traveling via a random walk on a circle with a finite number of stations, and that each station is equally likely to be the next destination, and that Yoda has devised a method which enables us to guess the next destination with overall probability greater than ½, as just described.

We embark upon the random walk, and after a sufficiently long time we are at an unknown station. If the random walk has been programmed BEFORE we began, and each move has already been determined, our next destination is determined by the flip of a coin. Under the above assumption, we can guess which direction we will go next with probability greater than ½. We are therefore guessing the result of a coin flip with probability greater than ½ - but that flip has already occurred.

But how is this different from a random walk that has not been programmed? When we reach the point where the train is equally likely to be in any station, it is easy to show that each station is equally likely to be the next destination. So – if there IS a next destination – we will be equally likely to be on either side of it, no matter where we are at the moment. The guessing procedure ONLY depends on making an observation from where we are, and we can do that BEFORE the coin is flipped.

And that's why I think this is even more puzzling than the Monty Hall Problem, because it would appear that we can guess the result of a coin flip before the coin is flipped – even if only as an overall average.

I'm happy to respond to any email – james.stein@csulb.edu